ON A METHOD OF COMPARISON FOR TRIPLE-SYSTEMS*

BY

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INTRODUCTION

1. If n elements $1, 2, 3, \cdots n$ can be distributed in triads in such a way that every pair of elements appears in one and only one triad, the totality of triads forms a triple-system of n elements. Reiss† has shown that it is possible to form a triple-system of n elements provided n is of the form $6m + 1$ or $6m + 3$.

Different methods for constructing triple-systems have been given by Reiss,† Netto,‡ Heffter,§ and E. H. Moore;‖ but methods for testing the non-congruency of these systems when formed are lacking. The group of substitutions that transform a triple-system into itself has hitherto been adopted as the abstract mark of the system, and Zulauf¶ has shown, by a consideration of the groups, that the four systems on 13 elements of Kirkman, Reiss, De Vries, and Netto are reducible to two incongruent systems whose groups are different. In the present paper it is shown however that for $n = 15$ two incongruent triple-systems, Δ_{15}, may have the same group.

By a simple process non-congruent triple-systems are constructed. These are used to illustrate a new method of comparison, by means of a new sort of abstract marks. This method requires no knowledge of the group but incidentally facilitates its determination. No exhaustive determination of every Δ_{15} which may be obtained by this process is here undertaken, but the 24 systems discussed include 12 apparently not hitherto constructed.

APPLICATION OF THE METHOD TO A REISS TRIPLE-SYSTEM ON 15 ELEMENTS

2. A triple-system formed by the Reiss method** is shown in the accompanying table, which gives 28 triples of the system; the remaining 7, formed

from the elements which head the columns, are abd, acg, aef, bce, bfg, cdf, deg.

TABLE I

a	b	c	d	e	f	g
87	86	85	84	83	82	81
62	71	76	75	74	73	72
53	52	31	61	65	64	63
41	43	42	32	21	51	54

This system may be arranged in the following rectangular array. Each element heads one column; below it are placed the 7 pairs of elements that occur with it in triads of the system.

a	b	c	d	e	f	g	1	2	3	4	5	6	7	8
87	86	85	84	83	82	81	$a4$	$a6$	$a5$	$a1$	$a3$	$a2$	$a8$	$a7$
62	71	76	75	74	73	72	$b7$	$b5$	$b4$	$b3$	$b2$	$b8$	$b1$	$b6$
53	52	31	61	65	64	63	$c3$	$c4$	$c1$	$c2$	$c8$	$c7$	$c6$	$c5$
41	43	42	32	21	51	54	$d6$	$d3$	$d2$	$d8$	$d7$	$d1$	$d5$	$d4$
bd	ad	ag	ab	af	ae	ac	$e2$	$e1$	$e8$	$e7$	$e6$	$e5$	$e4$	$e3$
cg	ce	be	cf	bc	bg	bf	$f5$	$f8$	$f7$	$f6$	$f1$	$f4$	$f3$	$f2$
ef	fg	df	eg	dg	cd	de	$g8$	$g7$	$g6$	$g5$	$g4$	$g3$	$g2$	$g1$

To examine the relations existing among the elements of the system, consider any triple, as $a87$ and the 21 pairs of elements which are united with the 3 elements of this triple.

a	8	7
87	$a7$	$a8$
62	$b6$	$b1$
53	$c5$	$c6$
41	$d4$	$d5$
bd	$e3$	$e4$
cg	$f2$	$f3$
ef	$g1$	$g2$

Begin with a pair in the column headed a, for example 62, written thus: call 6 the first element and 2 the second element of this pair; proceed from one column to the next in cyclical order, selecting in successive columns the pair which contains the second element of the preceding pair; continue this process until the pair to be selected from the first column is the initial pair. For the triple $a87$ the pairs would then appear as follows: 62, $2f$, $f3$, 35, $5c$, $c6$, 62, $2f$, the pairs repeating after each of the three columns has been examined twice. The occurring pairs, the periodicity, and the fact that the *period* consists of two rounds of the three columns can be shown conveniently by the notation $62f/35c/6$. The bar between two elements indicates the completion of one round of the three columns. Such a succession of pairs, repeating after 2 rounds of the columns, forms a *sequence* of period 2.

Since the elements in the pair 62 can be written either as 62 or 26, examine next this same pair with the two elements written in reverse order. Beginning with the pair 26 the pairs are selected from the columns as follows: 26, $6b$, $b1$; 14, $4d$, $d5$; 53, $3e$, $e4$; 41, $1g$, $g2$; 26, showing that after 4 rounds of the columns, the first column is again reached with the initial pair, or in sequence notation $26b/14d/53e/41g/2$.

Apply this process to the remaining pairs 53, 41, bd, cg, ef, of column a. The pair 53 appears in the sequence of period 4 and does not require further examination, since 53 as initial pair would furnish the sequence $53e/41g/26b$ $/14d/5$, which is identical with the above sequence of period 4 except as to starting point; hence any pair which has appeared in a sequence can be omitted from further examination. The pair 41 appears in the sequence of period 4; the pair bd furnishes the sequence $bd4/ef2/gc5/db6/cg1/b$, of period 5; the pair fe leads to the sequence $fe3/f$ of period 1; the pair 87 to the sequence $87a/8$. The pair 78 leads from the body of the first column up to the element 8 at the head of the second column and the process cannot be continued. Therefore in examining the pairs of a column we omit the pair which, with the element at the head of the column, completes the triple under examination.

For the triple $a87$ the sequences are $62f/35c/6$, $26b/14d/53e/41g/2$, $bd4/ef2/gc5/db6/cg1/b$, $fe3/f$; and all possible pairs for the column a, namely 62, 26, 53, 35, 41, 14, bd, db, cg, gc, ef, fe, appear in these sequences. The number of rounds of the columns required to close a sequence (the period) is recorded for each sequence belonging to the triple under examination, and we shall call these numbers the *index* of that triple, and the longest sequence the *major sequence.*

The index of the triple $a87$ consists of the numbers 2, 4, 5, 1. The index shows that the sequences for the triple $a87$ consist of a sequence closed in two cycles of three elements, hence containing two triads, a second sequence containing four triads, a third sequence containing five triads and a fourth sequence containing one triad. In any column twelve pairs must be considered and each pair occurs in one and only one round of the columns and therefore in one and only one triad in the sequences. Hence for a triple-system on 15 elements, the sum of the periods for every index is 12. Indices of the types 111117, 111144, 66 are written in the abbreviated form $1^5 7$, $1^4 4^2$, 6^2.

The sequences are closed cycles, in which any element may be selected as initial element and the sequence may be read in direct or reverse order. The sequences for $a87$, $87a$, and $8a7$ illustrate the following

THEOREM. *The index of every triple in a given triple-system is invariant under all permutations of its triad.*

ANALYSIS OF THE MAJOR SEQUENCES

3. An element which appears once, twice, three times in a sequence is designated as a single, double, triple point, respectively, for that sequence. It is evident that a sequence of period 1 contains no double point and hence that no duplicate element occurs in a triad of a sequence.

LEMMA. *A sequence of period greater than one contains no repetition in any quintad of elements.*

x	y	z
ab	bc	cd
de		

Let the sequence for xyz begin in the column x with the initial pair ab, and for convenience number the places in the sequence. The elements in places 1, 2, and 3 may be a, b, and c, but the element in the place 4 is not a, since that would close the sequence. No repetition occurs in a triad of a sequence, hence the element in place 4 is neither b nor c, and therefore must be a new element d. The element to fill place 5 forms with d a pair in column x and so is neither a nor b; since this required element forms a triad with c, d, it must be a new element e. Hence no repetition occurs in a quintad of elements. By applying the above Lemma and reading the sequence forward and backward we obtain the following

THEOREM. *A sequence of period equal to or less than three contains no multiple point.*

4. This process applied to the 35 triples of the Reiss Δ_{15} gives indices and sequences only of certain types, namely:

	TRIPLE	INDEX	SEQUENCES
(1)	abd	$1^5\,7$	$871/625/786/143/268/417/534/8$,
			$352/3$, cgf/c, gce/g, efg/e, fec/f;
(2)	$a87$	1245	$62f/35c/6$, $26b/14d/53e/41g/2$,
			$bd4/ef2/gc5/db6/cg1/b$, $fe3/f$;
(3)	$a41$	6^2	$87e/26f/53b/78d/62c/35g/8$,
			$bd8/gc2/ef6/db3/cg5/fe7/b$.

The indices have separated the 35 triples of the Reiss system into 3 classes, (1), (2), and (3). This separation into classes will be found to simplify the problem of determining the group of the system. For every substitution which converts the system into itself must permute among themselves the triples of a class, since formation of cycles depends on identity or difference of elements in the different columns and these are not changed by any substitution.

The 35 triples arranged in classes according to their indices are shown in the following table.

Classification of the Reiss Δ_{15} by indices

$1^5\,7$			1245							6^2		
abd	bce	deg	$a26$	$b25$	$c24$	$d23$	$e38$	$f28$	$g27$	$a14$	$d16$	$g18$
acg	bfg		$a35$	$b34$	$c58$	$d48$	$e47$	$f37$	$g36$	$b17$	$e12$	
aef	cdf		$a78$	$b68$	$c67$	$d57$	$e56$	$f46$	$g45$	$c13$	$f15$	
(1)			(2)							(3)		

In the major sequence 871/625/786/143/268/417/534/8 of the triple abd, the second 8 enters in the seventh place after the first 8, the third 8 in the seventh place after the second 8, and the first 8 in the seventh place after the third 8; we denote this by the symbol $8(7,7,7)$; with this notation the elements of the sequence are characterized as follows: $8(7,7,7)$, $7(5,11,5)$, $1(7,7,7)$, $6(5,5,11)$, $2(8,13)$, $5(13,8)$, $4(5,5,11)$, $3(8,13)$. For a sequence of period 7 containing 21 elements the symbols $(5,11,5)$, $(5,5,11)$, $(11,5,5)$ are identical and $(8,13) \equiv (13,8)$. Hence this sequence contains two distinct types of triple points and one type of double points.

The analysis of the major sequences of the triples of class (1) is given in the following table:

Class (1)	(7,7,7)	(5,11,5)	(7,7,7)	(5,5,11)	(8,13)	(13,8)	(5,5,11)	(8,13)
abd	8	7	1	6	2	5	4	3
bce	4	3	1	2	5	8	7	6
cdf	7	6	1	5	8	4	3	2
deg	3	2	1	8	4	7	6	5
efa	6	5	1	4	7	3	2	8
fgb	2	8	1	7	3	6	5	4
gac	5	4	1	3	6	2	8	7

DETERMINATION OF THE GROUP FOR THE REISS Δ_{15}

5. Any operation of the group that leaves a triple-system invariant can only transform any triad and its sequences into itself, or into another of the same class. Therefore to find the group, we examine first for substitutions that transform into itself one of the longest sequences, and secondly for those that transform it into the remaining sequences of its class. Substitutions when determined must be tested on the whole triple-system.

An examination of the table of indices for this system shows that the 15 elements do not enter symmetrically as members of the triads in the classes, for example, the 7 triads in class (1) contain only the elements a, b, c, d, e, f, g. Since only those elements may be permuted which occur the same number of times in a class, the enumeration of the appearances of each element in the classes, as in the following table, shows the possible sets of transitive elements.

	a	b	c	d	e	f	g	1	2	3	4	5	6	7	8
(1)	3	3	3	3	3	3	3								
(2)	3	3	3	3	3	3	3		6	6	6	6	6	6	6
(3)	1	1	1	1	1	1	1	7	1	1	1	1	1	1	1

The systems of transitivity for the group are therefore a, b, c, d, e, f, g; 1; $2, 3, 4, 5, 6, 7, 8$.

The substitutions may be determined from any class in the system, but most easily from the class containing the longest sequences since these exhibit

more repetitions of the elements. In the present case class (1) is therefore selected.

(1) Examine for substitutions to transform the triple *abd* into itself. The major sequence of *abd* is 8716257861432684175347534, and a consideration of the possible rotations of this sequence into itself will determine the substitutions which transform the triple *abd* into itself. The element 1 forms a transitive system; hence a rotation of period 3 given by (1) (647) (235) (8) transforms this sequence into itself. The triple-system will now enable us to complete the substitution, for if two elements in one triple transform into two elements in a second triple, the third element in the first triple transforms into the third element in the second triple, for example, 18*g* transforms into 18*g*. An application of both cycles (235) and (647) to the two triples 26*a* and 24*c* gives respectively 26*a*, 34*b*, 57*d*; 24*c*, 37*f*, 56*e* and shows for the required substitution additional cycles (*abd*) and (*cfe*). Hence the substitution (*abd*) (*cfe*) (*g*) (1) (235) (476) (8) transforms *abd* and its sequences into itself.

Any rotation of period 2 which transforms the sequence of *abd* into itself must keep fixed one of the elements 1 and interchange elements at equal distances from this 1, for example, (1) (46) (38) (27) $\cdots$, but in the sequence 3 is a double point and 8 is a triple point and it is not possible to convert the sequence into itself by the interchange of two points of different multiplicities; hence no substitution of period 2 exists which will convert this sequence into itself. Therefore only a sub-group of order 3 transforms the triple *abd* into itself.

(2) Examine for substitutions to transform the triple *abd* into the other triples in its class. The table of class (1) exhibits the cycles (8473625) (1). If now we apply these two cycles to any triple in the system, as *a*14, then the triples of the system appear in the following succession: 14*a*, 17*b*, 13*c*, 16*d*, 12*e*, 15*f*, 18*g*, showing the cycle (*abcdefg*) and this completes the substitution $s_2 \equiv$ (*abcdefg*) (1) (2584736) which transforms *abd* in succession into every other triple in its class. Therefore the group for the Reiss system is of order 21 and is generated by

$s_1 \equiv$ (*abd*) (*cfe*) (*g*) (1) (235) (476) (8), $s_2 \equiv$ (*abcdefg*) (1) (2584736).

The method of construction employed by Reiss involves a cyclic substitution of order 7, hence the appearance of s_2 was to be expected; but it did not show the substitution of order 3.

CONSTRUCTION OF NEW SYSTEMS

6. The seven triples containing only the elements which head the columns in the seven-by-four array, Table I, §2, form a triple-system on 7 elements, which is designated as a *head* for this array. The seven triples in the head

and the twenty-eight triples in the array make up the thirty-five triples in the system. New systems may be formed by retaining the twenty-eight triples of the seven-by-four array and changing the seven triples of the head or by changing the seven-by-four array. An index determined from a seven-by-four array is designated as a *partial* index to distinguish it from a complete index determined from a fifteen-by-seven array.

FORMATION OF NEW SYSTEMS BY CHANGING THE HEAD OF THE SEVEN-BY-FOUR ARRAY

7. From the seven elements a, b, c, d, e, f, g thirty-five triples can be formed. We determine for each of these triples its partial index from the seven-by-four array I. For example:

TRIPLE	INDEX	SEQUENCES
abc	1^2, 2, 4	786/7, 871/352/417/625/8, 268/534/2, 143/1;
abd	1, 7	871/625/786/143/268/417/534/8, 352/3.

Table of partial indices

1^2, 2, 4							1, 7				
abc	ace	adf	bcd	bdg	cde	cfg	abd	acg	bde	aef	deg
abe	acf	adg	bcf	bef	cdg	efg	abf	bce	cdf	aeg	dfg
abg	ade	afg	bdf	beg	ceg	def	acd	bcg	cef	bfg	

Select any triple, as abd. Construct all the triple-systems on the seven elements a, b, c, d, e, f, g, which contain abd, and retain one of each type. Six systems Δ_7 can be formed, but the sets of partial indices of four of those systems are identical. Hence this triple furnishes only three types of head for the seven-by-four array, namely A, B, and C, as in the table.

A		B		C	
abd	1, 7	abd	1, 7	abd	1, 7
acg	1, 7	acg	1, 7	ace	1^2, 2, 4
aef	1, 7	aef	1, 7	afg	1^2, 2, 4
bce	1, 7	bcf	1^2, 2, 4	bcf	1^2, 2, 4
bfg	1, 7	beg	1^2, 2, 4	beg	1^2, 2, 4
cdf	1, 7	dce	1^2, 2, 4	cdg	1^2, 2, 4
deg	1, 7	dfg	1, 7	def	1^2, 2, 4

Since triples with different partial indices may give different types of heads, we examine in the same way the triple abe; this furnishes 4 heads of type C and 2 heads of type B.

The 35 triples formed of the elements a, b, c, d, e, f, g contain only 2 partial indices, 1^2 24 and 17, and all triple-systems of seven elements have been formed for one triple selected from each class of partial indices, and three types of heads A, B, and C have appeared. The twenty-eight triples in the seven-by-four array completed with any one of these heads form a triple-system on 15 elements.

The array, Table I, completed by these three heads, is denoted by IA, IB, IC; of these IA is the Reiss system already investigated.

	a	b	c	d	e	f	g
I	87	86	85	84	83	82	81
	62	71	76	75	74	73	72
	53	52	31	61	65	64	63
	41	43	42	32	21	51	54
IA	abd	acg	aef	bce	bfg	clf	deg
IB	abd	acg	aef	bcf	beg	cde	dfg
IC	abd	ace	afg	bcf	beg	cdg	def

To show that the 3 systems IA, IB, IC, are non-congruent, it is necessary to write each system in the form of the fifteen-by-seven array and to obtain the indices for the 35 triples of each system; since it is possible that triples with different partial indices may have the same complete index and *vice versa*.

No two congruent systems have been admitted, but it does not follow that non-congruent systems have not been excluded, and in an exhaustive determination of all systems Δ_{15} every head must be applied to the seven-by-four array, and the indices for each system determined from its fifteen-by-seven array.

8. CLASSIFICATION OF IB BY INDICES AND TRANSITIVE ELEMENTS

$1^6\,24$	$1^5\,7$	$1^3\,234$	$1^2\,2^3\,4$	$1^2\,46$	1245			$2^3\,6$	$2^2\,4^2$		6^2
bcf	abd	d23	b25	b17	a87	b68	f28	d16	b34	d48	a14
beg	acg	f37	c67	c13	a26	e56	g36	f15	c24	f46	
cde	aef	g27	e38	e12	a35	c58	d57	g18	e47	g45	
	dfg										
(1)	(2)	(3)	(4)	(5)	(6)			(7)	(8)		(9)

The enumeration of the entrances of the elements in the 9 classes shows that for the group of the system the sets of transitive elements are a; b, c, e; d, f, g; 1; 2, 3, 7; 5, 6, 8; 4. These sets of transitive elements subdivide the classes into sets of triples which are not transformable into one another by operations of the group of the system; the subdivisions are shown by lines separating the triples in a class, and the system IB contains 13 subdivisions. The group is generated by $s \equiv (a)\,(bec)\,(dfg)\,(1)\,(237)\,(586)\,(4)$, and is of order 3.

9. CLASSIFICATION OF IC BY INDICES AND TRANSITIVE ELEMENTS

$1^6\,24$		$1^5\,7$	$1^3\,234$	$1^2\,2^3\,4$		$1^2\,46$		1245		$2^2\,4^2$	$2^3\,4$	6^2
ace	beg	abd	c58	a35	g27	a14	c13	a78	c24	a26	c67	g18
def	cdg		e38	b25	g36	b17	e12	b68	e56	b34	e47	
bcf	afg		f28	d23	g45	d16	f15	d48	f37	d57	f46	
(1)		(2)	(3)	(4)		(5)		(6)		(7)	(8)	(9)

The sets of transitive elements for the system are a, b, d; c, e, f; g; 1; 2, 3, 5; 4, 6, 7; 8. These with the indices separate the system into 13 non-permutable subdivisions. The group is generated by $s \equiv (adb)\,(cef)\,(g)\,(1)\,(253)\,(467)\,(8)$, and is of order 3.

10. A comparison of IB and IC shows that the groups of these two non-congruent systems are the same. For these groups are generated respectively by $s \equiv (a)\,(bec)\,(dfg)\,(1)\,(237)\,(4)\,(586)$, $t \equiv (adb)\,(cef)\,(g)\,(1)\,(253)\,(467)\,(8)$, and are abstractly the same, since each of these substitutions permutes 12 elements, in 4 cycles, each of order 3, and leaves the remaining 3 elements fixed. But the system IB contains 4 triples with index $1^5\,7$, while the system IC contains only 1 triple with index $1^5\,7$. Now in a given system the index of a triple is invariant under all the substitutions of the symmetric group; two systems are, therefore, certainly incongruent if their sets of indices are unlike, whether their groups are different or identical.* *Hence no substitution exists which converts IB into IC, and the two systems are non-congruent. That is, as pointed out in §1, the same group belongs to two non-congruent systems.*

Up to the present time the group has been considered as the abstract mark of a system. It is true that two systems with different groups are distinct, but as shown above, the converse is not true, a fact which apparently has not been suspected before.

FORMATION OF NEW SYSTEMS BY CHANGING THE FUNDAMENTAL SEVEN-BY-FOUR ARRAY

11. If a seven-by-four array shows 3 columns of the types x, y, and z containing the 3 sets of pairs 12, 34: 13, 24: 14, 23 which can be formed from the four elements 1, 2, 3, 4, the four elements constitute a *quadrangular array* which may be denoted by the symbol $\left[{}^{12}_{34}\right]$. The remaining

x	y	z
12	13	14
34	24	23
56	57	58
78	68	67

four elements 5, 6, 7, 8 must necessarily form also a quadrangular array; hence quadrangular arrays enter into a seven-by-four array always by couples, and it is easily shown that a seven-by-four array must contain 0, 2, 6, or 14 quadrangular arrays; — a fact hitherto not noticed, but of considerable importance in the following discussion.

An examination of every combination of two pairs occurring in the 7 columns of Table I shows that this seven-by-four array contains no quadrangular array.

Let us form, then, a new fundamental Table II, which shall contain two and only two of these quadrangular arrays. In constructing columns suitable for a seven-by-four array, the first pair in the column may be selected in 7 ways (since with the first element we may unite any one of the remaining

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* The truth of the converse is not here asserted, though it seems probable.

7 elements), the second pair in 5 ways, the third pair in 3 ways, and the fourth pair only in 1 way; hence 105 suitable columns may be formed.

From the 105 columns we select 3 columns a, b, c, containing the 2 quadrangular arrays $[{}^1_3{}^2_4]$ and $[{}^5_7{}^6_8]$. The remaining 4 columns d, e, f, g, must be selected so that no more quadrangular arrays are introduced, and the entering columns must not contain any pair that appears in the columns already in the table.

<div align="center">TABLE II</div>

a	b	c	d	e	f	g
87	86	85	84	83	82	81
65	75	76	73	72	71	74
43	42	41	61	64	63	62
21	31	23	25	15	45	53

The Table II shows one of the several possible ways of constructing a seven-by-four array of the desired type. The determination, from the seven-by-four array II, of the partial indices for the 35 triples which may be formed from the 7 elements a, b, c, d, e, f, g, shows 5 different partial indices. The re-heading process is now applied to Table II. A triple of each index furnishes 6 systems Δ_7. By the selection of 1 system of each type from these 30, the 6 different heads A, B, C, D, E, F shown in the following table are obtained for the seven-by-four array II.

	a	b	c	d	e	f	g	NUMBER OF TRIPLES OF THE HEAD WITH PARTIAL INDICES				
II	87 65 43 21	86 75 42 31	85 76 41 23	84 73 61 25	83 72 64 15	82 71 63 45	81 74 62 35	1^8	$1^4\,2^2$	$1^2\,24$	2^4	4^2
A	bde	abc	bfg	adf	cdg	aeg	cef	1	0	4	2	0
B	bde	abf	bcg	acd	dfg	aeg	cef	0	0	2	2	3
C	bde	abf	bcg	adg	cdf	ace	efg	0	2	1	1	3
D	bde	abg	bcf	acd	dfg	aef	ceg	0	2	1	2	2
E	bde	bac	bfg	dag	cdf	aef	ceg	1	4	2	0	0
F	bde	abg	bcf	adf	cdg	ace	efg	0	0	2	3	2

If a system Δ_{15} contains more than one Δ_7, then for every such contained Δ_7 there is an arrangement of the Δ_{15}, showing this Δ_7 as a head to a suitable seven-by-four array. These different arrangements may furnish dissimilar partial indices. For example, the system IIA contains three systems Δ_7 formed of the elements a, b, c, d, e, f, g; a, b, c, 1, 2, 3, 4; and a, b, c, 5, 6, 7, 8. The second of these heads, placed over the seven-by-four array formed of the 28 remaining triples of the system IIA, gives for the triples abc, $a43$, $a12$, $b42$, $b31$, $c41$, $c23$, respectively, the following partial indices 1^8, $1^4\,2^2$, $1^4\,2^2$, 2^4, 2^4, 4^2, 4^2. Hence IIA shows dissimilar sets of partial indices for different heads. Therefore if a system Δ_{15} contains more than one Δ_7, a determination of the *partial indices* of the seven triples of one head is not a sufficient characterization of the system.

To determine if the six systems with fundamental Table II are distinct, we compare the indices of the thirty-five triples of each system, since systems with different sets of indices are incongruent. The indices for the 6 systems with fundamental seven-by-four array II are shown in the following table:

II	1^{12}	$1^8 2^2$	$1^6 24$	$1^5 2^2 3$	$1^4 2^4$	$1^4 26$	$1^4 4^2$	$1^3 2^2 5$	$1^3 234$	$1^2 2^3 4$	$1^2 2^2 3_2$	$1^2 46$	1236
A	1		12		10						12		
B			2		2	4	3	4	4	4	8		4
C		2	1	6	5		3	4	4		4	2	4
D		2	1	6	4		2	6	2	2	8	2	
E	1	10	4	8	6		2			4	4		
F			2		5	4	2			4	10		4

This table shows 6 distinct systems differing from the 3 systems with fundamental seven-by-four array I.*

The sets of transitive elements for IIA are a, b, c; d, f, 2, 4, 5, 8; e, g, 1, 3, 6, 7; these with the classes separate the 35 triples of the system into 6 non-permutable su'divisions. The group is generated by $s_1 \equiv (13)(24)(58)(67)$, $s_2 \equiv (ab)(d2f4)(e1g3)(58)$, $s_3 \equiv (abc)(d48)(e17)(f25)(g36)$, and is of order 24.

The sets of transitive elements for IIB are a; b, c; d, f; e; g; 1, 3, 6, 7; 2, 4, 5, 8; these with the classes separate the system into 13 non-permutable subdivisions. The group is generated by $s_1 \equiv (bc)(1637)(2548)(df)$, and is of order 4.

The sets of transitive elements for IIC are a; b; c; d; e; f; g; 1, 3; 2, 4; 5, 8; 6, 7; these with the classes separate the system into 23 non-permutable subdivisions. The group is generated by $s_1 \equiv (13)(24)(58)(67)$, and is of order 2.

The sets of transitive elements for IID are a; b; c; d; e; f; g; 1, 3; 2, 4; 5, 8; 6, 7; these with the classes separate the system into 23 non-permutable subdivisions. The group is generated by $s \equiv (13)(24)(58)(67)$, and is of order 2.

The two systems IIC *and* IID *are distinct, since the number of triples appearing under each index is not the same for the two systems; but the group is identical for the two systems.* This is another example of the fact that two distinct systems may have the same group.

The sets of transitive elements for IIE are a; b; c; d, f, 6, 7; e, g, 5, 8; 1, 3; 2, 4; these with the classes separate the system into 12 non-permutable subdivisions. The group is generated by $s_1 \equiv (13)(24)(58)(67)$, $s_2 \equiv (d7f6)(e5g8)(13)(24)$, and is of order 8.

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* For the sake of brevity the work involved in the determination of the indices for every system is suppressed, and a brief outline of the process employed in obtaining the groups is given only for the system IA. A fuller elaboration is contained in a manuscript volume deposited in the Vassar College Library.

The sets of transitive elements for IIIF are a; b, c; d; f; e, g; 1, 3, 6, 7; 2, 4, 5, 8; these with the classes separate the system into 15 non-permutable subdivisions. The group is generated by $s_1 \equiv (13)(24)(58)(67)$, $s_2 \equiv (bc)(eg)(17)(28)(36)(45)$, and is of order 4.

SEVEN-BY-FOUR ARRAY WITH 14 QUADRANGULAR ARRAYS

12. We construct next a seven-by-four array containing 14 quadrangular arrays, the greatest possible number, and we apply to this new array the re-heading process. The partial indices show 4 types of heads and we obtain the following systems:

	a	b	c	d	e	f	g	NUMBER OF TRIPLES WITH INDICES				
								1^{12}	$1^8 2^2$	$1^6 3^2$	$1^4 2^4$
III	12	13	14	15	16	17	18					
	34	24	23	26	25	28	27					
	56	58	57	38	37	36	35					
	78	67	68	47	48	45	46					
A	abc	ade	afg	bdg	bef	cdf	ceg	35			
B	abc	ade	afg	bdf	beg	cdg	cef	7	24		4
C	abc	adg	aef	bdf	beg	cde	cfg	1	12	16	6
D	abd	acg	aef	bcf	beg	cde	dfg			28	7

The system IIIA contains one set of transitive elements a, b, c, d, e, f, g, 1, 2, 3, 4, 5, 6, 7, 8, and the thirty-five triples belong to one class.

The analysis of the Kirkman system by the method of sequences shows the index of each triple to be 1^{12}, and the system IIIA is congruent to the Kirkman system. The group for the system IIIA is by far the most extensive of the 24 groups determined in this paper, and belongs to the earliest known Δ_{15}.

The group for IIIA is generated by $s_1 \equiv (12)(34)(56)(78)$, $s_2 \equiv (13)(24)(58)(67)$, $s_3 \equiv (15)(26)(38)(47)$, $s_4 \equiv (acb)(efg)(243)(678)$, $s_5 \equiv (bc)(fg)(34)(78)$, $s_6 \equiv (bc)(dfeg)(34)(5768)$, $s_7 \equiv (adgfceb)(1347286)$, $s_8 \equiv (ad7gfc5368b1e42)$, and is of order 8!/2.*

The sets of transitive elements for IIIB are a; b, c, d, e, f, g; 1, 2, 3, 4, 5, 6, 7, 8. The group for the system is generated by $s_1 \equiv (beg)(cdf)(12)(358467)$, $s_2 \equiv (df)(eg)(57)(68)$, $s_3 \equiv (15)(26)(38)(47)$, $s_4 \equiv (de)(fg)(56)(78)$, and is of order 96.

The sets of transitive elements for IIIC are a, b, c; d, e, f, g; 1, 2, 3, 4, 5, 6, 7, 8. The group for the system IIIC is generated by $s_1 \equiv (abc)(dfg)(234)(578)$, $s_2 \equiv (16)(25)(37)(48)$, $s_3 \equiv (12)(34)(56)(78)$, $s_4 \equiv (dg)(ef)(58)(67)$, and is of order 96. The groups for the systems IIIB and IIIC are of the same order 96, but the sets of transitive elements for the two groups differ, hence the groups are distinct.

* C. Jordan (*Traité des substitutions et des équations algébriques* (1870), p. 380) states that the group for the Kirkman system is of order 8!/2, which agrees with the result obtained by the sequence method of analysis.

The sets of transitive elements for IIID are a, b, c, d, e, f, g; 1, 2, 3, 4, 5, 6, 7, 8. The group is generated by $s_1 \equiv (acg)(bef)(248)(367)$, $s_2 \equiv (15)(26)(38)(47)$, $s_3 \equiv (abedgcf)(3658472)$, $s_4 \equiv (13)(24)(58)(67)$, and is of order 168.

Seven-by-four array with six quadrangular arrays

13. We construct finally a seven-by-four array containing 6 quadrangular arrays, the only remaining number (cf. §11) and we obtain the 4 following systems:

	a	b	c	d	e	f	g
IV	87	86	85	84	83	82	81
	65	75	76	73	74	71	72
	43	41	42	62	61	64	63
	21	23	31	15	25	53	54
A	bdf	abc	beg	adg	cde	afe	cfg
B	bdf	abe	bcg	adg	cde	acf	efg
C	bdf	abe	cef	bcg	acd	deg	afg
D	bdf	abc	beg	ade	cdg	afg	ecf

IV	1^{13}	$1^8 2^2$	$1^4 24$	$1^6 3^2$	$1^5 2^2 3$	$1^4 2^4$	$1^4 26$	$1^4 4^2$	$1^2 2^3 3^2$
A	1	8				16	10		
B		1		4		8	6	2	6
C	1	10	4			8	6	2	4
D	7	24					4		

The systems IVC and IVD show the same indices as the systems IIE and IIIB, respectively, and are congruent to them.

The sets of transitive elements for IVA are a, b; c; d, e, f, g, 1, 2, 3, 4; 5, 6, 7, 8; and the system contains 7 non-permutable subdivisions. The group for the system is generated by $s_1 \equiv (ab)(d1)(e3)(f2)(g4)(67)$, $s_2 \equiv (ab)(d2f1)(e4g3)(5687)$, $s_3 \equiv (13)(24)(57)(68)$, and is of order 32.

The systems of transitive elements for IVB are a; b; c; d; e; f; g; 1, 2, 3, 4; 5, 6, 7, 8; and the system separates into 17 non-permutable subdivisions. The group is generated by $s_1 \equiv (12)(34)(56)(78)$, $s_2 \equiv (14)(23)(58)(67)$, and is of order 4.

Formation of new systems by varying the number of partial quadrangular arrays

14. The three sets of pairs 12, 34; 13, 24; 14, 23 entering the seven-by-four array in 3 columns form a quadrangular array which has been denoted by the symbol $\left[\begin{smallmatrix}12\\34\end{smallmatrix}\right]$. Two of these sets of pairs occurring in the seven-by-four array in 2 columns form a *partial* quadrangular array, and may be denoted by the symbol $\left\{\begin{smallmatrix}12\\34\end{smallmatrix}\right\}$. A partial quadrangular array considered with respect to the two elements which head the columns in which the two sets of pairs enter, is designated in a recent paper as an " interlacing "* of the two ele-

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* F. N. Cole, *The triad systems of thirteen elements.* These T r a n s a c t i o n s, vol. 14 (1913), pp. 1–5.

ments which head the columns. Non-congruent triple-systems may be formed by keeping constant the number of quadrangular arrays in the seven-by-four array, and varying the number of partial quadrangular arrays.

The seven-by-four array II contains two quadrangular arrays $[^{12}_{34}]$, $[^{56}_{78}]$ and eight partial quadrangular arrays $\{^{12}_{56}\}$, $\{^{12}_{78}\}$, $\{^{13}_{58}\}$, $\{^{24}_{67}\}$, $\{^{24}_{58}\}$, $\{^{24}_{67}\}$, $\{^{34}_{56}\}$, $\{^{34}_{78}\}$. The seven-by-four array V, given below, contains the same number of quadrangular arrays, $[^{12}_{58}]$, $[^{34}_{67}]$, but a different number of partial quadrangular arrays, the twelve $\{^{12}_{46}\}$, $\{^{12}_{37}\}$, $\{^{13}_{48}\}$, $\{^{13}_{56}\}$, $\{^{14}_{57}\}$, $\{^{16}_{78}\}$, $\{^{23}_{45}\}$, $\{^{23}_{68}\}$, $\{^{24}_{78}\}$, $\{^{25}_{67}\}$, $\{^{35}_{78}\}$, $\{^{45}_{68}\}$. The re-heading process applied to this seven-by-four array shows six types of heads.

V	a	b	c	d	e	f	g	NUMBER OF TRIPLES WITH PARTIAL INDICES				
	12	15	18	14	17	13	16			1^3	$1^4 2^2$	2^4	4^2
	58	28	25	26	23	27	24						
	34	37	36	38	45	48	35						
	67	46	47	57	68	56	78						
A	abc	ade	afg	bdf	beg	cdg	cef			1		6	
B	abd	ace	afg	bcf	beg	cdg	def					4	3
C	abd	ace	afg	bcg	bef	cdf	deg				2	2	3
D	abd	acf	aeg	bcg	bef	cde	dfg				3	1	3
E	abc	ade	afg	bdg	bef	cdf	ceg			1	4	2	
F	abc	adf	aeg	bdg	bef	cde	cfg			1	6		

A determination of the indices of the 6 systems with fundamental seven-by-four array V gives the following result:

V	1^{12}	$1^8 2^2$	$1^6 3^2$	$1^5 2^2 3$	$1^4 2^4$	$1^4 35$	$1^4 4^2$	$1^3 234$	$1^3 3^3$	$1^3 2^2 3^2$
A	1				18				16	
B					4		3	12	4	12
C		2		4	6	8	3	4		8
D		3	4		7	12	3			6
E	1	8		16	10					
F	1	12	16		6					

The triple-systems VE and VF show the same sets of indices as IVA and $IIIC$, respectively, and are congruent to them. The analysis follows for the 4 systems VA, VB, VC, VD, which are distinct from the preceding 15 triple-systems.

The sets of transitive elements for VA are $a, b, c; d, e, f, g, 1, 2, 3, 4, 5, 6, 7, 8$. The group is generated by $s_1 \equiv (abc)(efg)(258)(367)$, $s_2 \equiv (bc)$ $(d1)(e2)(f8)(g5)(67)$, $s_3 \equiv (d14)(e23)(f56)(g87)$, $s_4 \equiv (acb)(dfg)$ $(182)(376)$, and is of order 288.

The sets of transitive elements for VB are $a, b, c; d, e, f; g; 1, 2, 3, 4, 5, 6, 7, 8$; these with the classes separate the system into 6 non-permutable subdivisions. The group is generated by $s_1 \equiv (ab)(ef)(1327)(4865)$, $s_2 \equiv (ab)(ef)(14)(26)(35)(78)$, $s_3 \equiv (acb)(def)(182)(467)$, and is of order 24.

The sets of transitive elements for VC are a; b, c; d, e; f; g; 1, 2, 3, 4, 5, 6, 7, 8; these with the classes separate the system into 11 non-permutable subdivisions. The group for the system is generated by $s_1 \equiv (bc)(de)(16)(27)(35)(48)$, $s_2 \equiv (bc)(de)(1457)(2386)$, and is of order 8.

The sets of transitive elements for VD are a, b, c; d, f, g; e; 1, 2, 5, 8; 3, 4, 6, 7; these with the classes separate the system into 7 non-permutable subdivisions. The group for the system is generated by $s_1 \equiv (12)(37)(46)(58)$, $s_2 \equiv (18)(25)(34)(67)$, $s_3 \equiv (acb)(dfg)(125)(347)$, and is of order 12.

ANALYSIS OF TRIPLE-SYSTEMS FORMED BY OTHER METHODS

15. A few of the well-known systems Δ_{15} are now examined, in order to determine, if possible, systems incongruent to the 19 already given.

(i) *Analysis of the Carpmael systems.* – The indices for the systems given by Carpmael[*] show two non-congruent systems. One of these yields a set of indices identical with that of VA; hence this system is omitted from further examination. The other Carpmael system yields a new set of indices; therefore we determine in this system a Δ_7, arrange the system in the form of a seven-by-four array, and apply the re-heading process, thus obtaining four systems incongruent to the preceding 19, VIA, VIB, VIC, and VID of the following table:

	a	b	c	d	e	f	g							
VIA	12	17	16	15	18	14	13	abe	acf	adg	bdf	cde	efg	bcg
VIB	34	23	24	28	27	25	26	abc	adg	aef	bde	bfg	cdf	ceg
VIC	58	48	38	36	35	37	45	abd	acg	aef	bce	bfg	cdf	deg
VID	67	56	57	47	46	68	78	abg	ace	adf	bcd	bef	cfg	deg

A comparison of the two seven-by-four arrays I and VI shows that neither contains any quadrangular array, but that while I contains no partial quadrangular array, VI contains the following six $\{^{12}_{58}\}$, $\{^{14}_{68}\}$, $\{^{15}_{67}\}$, $\{^{23}_{48}\}$, $\{^{23}_{67}\}$, and $\{^{24}_{67}\}$. Hence the two sets of systems I and VI may be formed by constructing seven-by-four arrays, which contain a fixed number (zero) of quadrangular arrays and a variable number of partial quadrangular arrays. The analysis and the groups for the system VI are given below.

	$1^2 2^2$	$1^2 24$	$1^2 2^3$	$1^4 7$	$1^2 2^4$	$1 \cdot 35$	$1 \cdot 4^4$	$1^2 2^5$	$1 \cdot 234$	$1^2 2^4$	$1^2 2^{3^2}$	$1 \cdot 5^5$	1236	1245	$13^2 5$	$2 \cdot 6$	2^5
VIA				4	3			6					12		6		4
VIB	1	2		2	1	2	1	3	2	2	8	2	2	4	1	2	4
VIC		3		1			3	3		3	6	9	6	3			1
VID	3	3	6	1		3		3		3	6	6					1

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* *Some solutions of Kirkman's 15-school-girl problem*, Proceedings of the London Mathematical Society, vol. 12 (1881), pp. 148–156.

The sets of transitive elements for VIA are a, b, c, d, e, f; g; 1, 2, 4, 7; 3, 5, 6, 8. The group is generated by $s_1 \equiv (abe)\,(cfd)\,(172)(386)$, $s_2 \equiv (ad)\,(bc)\,(14)\,(27)\,(35)\,(68)$, $s_3 \equiv (bc)\,(ef)\,(12)\,(36)(47)\,(58)$, and is of order 12.

The sets of transitive elements for VIB are a; b, c; d; e, f; g; 1, 2; 3, 6; 4, 7; 5, 8. These with the classes separate the system into 21 non-permutable subdivisions. The group is of order 2 and is generated by $s \equiv (bc)\,(ef)\,(12)\,(36)\,(47)\,(58)$.

The sets of transitive elements for VIC are a, b, e; c, d, f; g; 1, 2, 7; 4; 5; 3, 6, 8. These with the classes separate the system into 13 non-permutable subdivisions. The group for the system is generated by $s \equiv (aeb)\,(cdf)\,(127)\,(368)$, and is of order 3.

The sets of transitive elements for VID are a, c, e; b, d, f; g; 1, 4, 7; 2; 6; 3, 5, 8. These with the classes separate the system into 13 non-permutable subdivisions. The group for the system is generated by $s \equiv (ace)\,(bfd)\,(147)\,(358)$, and is of order 3.

A comparison of the systems IB, IC, VIC, VID, shows different sets of indices but identical groups. *Here then are four incongruent systems with the same group* (cf. §8 and §9). *Therefore it is evident that the identity of the groups, though a necessary condition for the congruency of two triple-systems, is not sufficient.*

(ii) *Analysis of the Heffter systems.* — The Heffter[*] method of constructing triple-systems of $6m + 3$ elements furnishes two systems for $6m + 3 = 15$. The first of these contains 15 systems Δ_7, the index of each triple is 1^{12}, and the system shows the same set of indices as the system here denoted by IIIA. The second system contains no Δ_7, and is therefore distinct from the preceding 23 systems, each of which contains at least one Δ_7. This second system is now denoted by VII. Its analysis is shown in the following table:

Classification of VII by indices

4, 8		3, 9									
$af4$	$d27$	$a58$	$ag7$	$bc7$	$b24$	$ce4$	$df5$	$eg6$	$f17$	$g28$	367
$bg5$	$e38$	ade	$ac2$	bef	$cd8$	$c35$	$d46$	$e57$	$f68$	145	478
$c16$		$ab6$	$a13$	$b18$	cfg	$dg1$	$e12$	$f23$	$g34$	256	$bd3$

The system contains 1 set of transitive elements: a, b, c, d, e, f, g, 1, 2, 3, 4, 5, 6, 7, 8, and separates into 2 non-permutable subdivisions. The group is generated by $s_1 \equiv (ad63)\,(c8f2)\,(e471)$, $s_2 \equiv (abcdefg12345678)$, and is of order 60.

(iii) Netto[†] describes two methods for constructing triple-systems of 15

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[*] L. Heffter, *Ueber Tripelsysteme*, M a t h e m a t i s c h e A n n a l e n, vol. 49 (1897), pp. 101–112.

[†] L. c.

elements. The analysis of the two systems by the method of sequences shows that the index of every triple in each system is 1^{12} and the two systems are of the type here designated by IIIA.

(iv) E. H. Moore* gives different methods for constructing triple-systems. A system formed by one of these methods was found to be of the type IIIA. Under the re-heading process a system formed by a second method yielded 4 systems congruent, respectively, to the systems here designated as IIE, IIIB, IIIC, IVA.

16. We have shown for the two fundamental seven-by-four arrays containing, respectively, zero and two quadrangular arrays, that non-congruent systems may be formed by varying the number of partial quadrangular arrays. It seems possible that additional systems Δ_{15}, each containing at least one Δ_7, might be formed by the same method from these two arrays, and also from the two fundamental arrays containing, respectively, six and fourteen quadrangular arrays. The purpose of this investigation is primarily to exhibit a new method of comparison for triple-systems, and while several systems Δ_{15} not hitherto constructed have been obtained, no exhaustive determination of all possible systems Δ_{15} has been undertaken.

The method of comparison here investigated is applicable to any system Δ_{2t+1} by merely arranging the system in the form of a $2t + 1$-by-t array and determining the indices for this array. In constructing systems Δ_{15} which contain no Δ_7, the number of partial quadrangular arrays admitted into a fifteen-by-seven array must be considered, but for systems Δ_{15} which contain one or more systems Δ_7, the number of complete and partial quadrangular arrays admitted into a seven-by-four array is to be considered. Exactly similar conditions exist in the case of systems Δ_{2t+1}. In constructing systems Δ_{2t+1} which contain no Δ_t, the consideration of a $2t + 1$-by-t array is necessary, but for systems Δ_{2t+1} which contain one or more systems Δ_t, the consideration of a t-by-$\frac{1}{2}(t + 1)$ array is sufficient.

April, 1913

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* L. c.